JOURNAL OF APPROXIMATION THEORY 37, 44–50 (1983)

Joint Approximation in the Unit Disc

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Communicated by E. W. Cheney

Received October 19, 1981

1. INTRODUCTION AND NOTATION

We shall consider various special cases of a general problem in joint approximation. Let A denote a space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. The set of all polynomials is denoted by P. We suppose that there is given a topology τ on A such that P is τ-dense in A. We write $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C}: |z| = 1\}$.

A relatively closed subset F of $\mathbb{D}$ is called a *Mergelyan set* for (A, τ) if the following property holds: given $f \in A$ such that the restriction $f|_F$ to F is uniformly continuous, there exists a sequence $\{p_\nu\}$ in P such that

(a) $p_\nu \to f$ uniformly on F,

(b) $p_\nu \to f$ in the topology τ.

A *Farrell set* for (A, τ) has a similar definition: given $f \in A$ such that $f|_F$ is bounded, then there exists $\{p_\nu\}$ such that

(α) $p_\nu \to f$ pointwise on F and

$$\lim_{\nu \to \infty} \sup\{|p_\nu(z)|: z \in F\} = \sup\{|f(z)|: z \in F\}$$

and such that (b) holds as above.

The problem is to describe Farrell and Mergelyan sets. In particular, one would like to obtain geometrical characterizations of such sets. This has already been done in the following two cases:

* The research was partially supported by a grant from the National Science Foundation.

0021-9045/83/010044-07$03.00/0

(A) A is the space H^∞ consisting of all bounded analytic functions on $\mathbb{D}$. The topology τ is that of pointwise bounded convergence on $\mathbb{D}$. (More precisely, τ is either the weak-star or the bounded weak-star topology on H^∞.) See [7, 8] for details and further references.

(B) A is the space $H(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$. Here τ denotes the topology of uniform convergence on compact subsets of $\mathbb{D}$. (See [7, 9, 10].)

This problem was introduced by the first author (see [5]). Recently [6] he raised the question of describing the Farrell and the Mergelyan sets for the Hardy spaces H^p. The object of this note is to answer this question and give a new characterization of Farrell and Mergelyan sets in terms of certain distance estimates to the space of continuous functions on $\bar{F}$ (the closure of F). A number of open problems are also discussed.

We remark that for all examples known so far, the classes of Farrell and Mergelyan sets coincide, and are also independent of p. It is a natural question to ask what the "reason" is, and in what greater generality this phenomenon persists.

2. Farrell and Mergelyan Sets for H^p

Let $\mathbb{T}$ denote the unit circle and m the normalized Lebesgue measure on $\mathbb{T}$. If $p < \infty$, then L^p consists of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_p = \left\{\int_{\mathbb{T}} |f|^p \, dm\right\}^{1/p} < \infty.$$

If $1 \leqslant p < \infty$, then L^p is a Banach space. The subspace H^p consists of all $f \in L^p$ whose Fourier coefficients with negative indices all vanish. We recall that if $f \in H^p$, then its harmonic extension to $\mathbb{D}$ (by means of the Poisson integral) is analytic on $\mathbb{D}$. For more details about H^p, see [3].

If B is a set and g is a function defined for each $z \in B$, we shall use the notation

$$\|g\|_B = \sup\{|g(z)| : z \in B\}.$$

If $z \in \mathbb{T}$ and $\{\zeta_\nu\} \subset \mathbb{D}$, we say that $\zeta_\nu \to z$ nontangentially if $\lim_{\nu\to\infty} |z - \zeta_\nu| = 0$ and if $|z - \xi_\nu|/(1 - |\xi_\nu|)$ remains bounded as $\nu \to \infty$.

We can now formulate our main theorem about H^p. Let τ_p denote the weak topology on H^p. This means that a sequence $\{f_\nu\} \subset H^p$ tends to $f \in H^p$ if and only if

$$\lim_{\nu\to\infty} \int_{\mathbb{T}} f_\nu g \, dm = \int_{\mathbb{T}} fg \, dm$$

for all $g \in L^q$, where $1/p + 1/q = 1$. (If $p = 1$, then L^q denotes the space L^∞ of all bounded measurable functions on $\mathbb{T}$.)

THEOREM. *Let $1 \leqslant p < \infty$, and let F be a relatively closed subset of $\mathbb{D}$. The following statements are equivalent:*

(i) *F is a Farrell set for (H^p, τ_p).*

(ii) *F is a Mergelyan set for (H^p, τ_p).*

(iii) *There is a set $E \subset \mathbb{T} \cap \bar{F}$ with $m(E) = 0$, such that if $\zeta \in \bar{F} \cap \mathbb{T} \backslash E$, then there is a sequence $\{\zeta_\nu\} \subset F$ converging nontangentially to ζ.*

(iv) *Assume g is a uniformly continuous function on F and that $f \in H^p$ has the property that $f|_F$ is bounded. Then there exist polynomials p_ν, $\nu = 1, 2, \ldots$, such that $p_\nu \to f$ in τ_p and $\lim_{\nu \to \infty} \| g - p_\nu \|_F = \| g - f \|_F$.*

Remark. Since condition (iii) is independent of p, we see that the Hardy spaces H^p $(1 \leqslant p < \infty)$ have the same Farrell as Mergelyan sets, and that their structure is independent of p. The characterization (iii) also holds for H^∞, as was shown in [7–10].

3. Proof of the Theorem

We show that (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii), and that (iii) $\Rightarrow$ (iv). That (iv) $\Rightarrow$ (i) follows by choosing $g = 0$ on F, and to see that (iv) $\Rightarrow$ (ii), we merely choose $g = f|_F$ if $f \in H^p$ and $f|_F$ is uniformly continuous.

Let us assume (iii) fails. To show that (i) fails, we recall from [8] that there is a function $f \in H^\infty$ such that

$$1 = \|f\|_{\mathbb{D}} = \text{ess sup}\{|f(e^{i\theta})| : e^{i\theta} \in \bar{F} \cap \mathbb{T}\}$$

and

$$\|f\|_F < 1.$$

Assume now that $\{p_\nu\} \subset P$, $p_\nu \to f$ in τ_p. Define $g \in L^q$ by

$$\begin{aligned} g &= \bar{f} \quad \text{on} \quad E = \{e^{i\theta} \in \bar{F} \cap \mathbb{T} : |f(e^{i\theta})| = 1\} \\ &= 0 \quad \text{on} \quad \mathbb{T} \backslash E, \end{aligned}$$

and remark that f can be constructed so that $m(E) > 0$. Now we have

$$\lim \int_{\mathbb{T}} p_\nu g = \int_E |f|^2 = m(E).$$

But since $\|p_\nu\|_F \to \|f\|_F < 1$, we also have

$$\limsup \left| \int_{\mathbb{T}} p_\nu g \right| \leqslant \limsup \|p_\nu\|_F \int_E |g| = \|f\|_F \, m(E),$$

and this is a contradiction. Hence (i) fails if (iii) fails.

To show that (ii) also fails, we use a construction due to Detraz [1, pp. 335–337]. If (iii) fails, she constructs a function $f \in H^\infty$ such that $f|_F$ is uniformly continuous and such that $f(z) \to 0$ as $z \to \zeta \in B$, $z \in F$, where B is a certain subset of $\bar{F} \cap \mathbb{T}$, with $m(B) > 0$. Again, if $p_\nu \to f$ in the τ_p topology and $\|p_\nu - f\|_F \to 0$ as $\nu \to \infty$, we shall have a contradiction. We may assume $f(0) \neq 0$. Since $p_\nu(0) \to f_\nu(0)$, Jensen's inequality

$$\log |p_\nu(0)| \leqslant \int_{\mathbb{T}} \log |p_\nu| \, dm$$

leads to a contradiction as follows:

$$\int_{\mathbb{T}} \log |p_\nu| \, dm = \int_B \log |p_\nu| \, dm + \int_{\mathbb{T} \setminus B} \log |p_\nu| \, dm.$$

But

$$\int_{\mathbb{T} \setminus B} \log |p_\nu| \, dm \leqslant \int_{\mathbb{T} \setminus B} |p_\nu| \, dm \leqslant \int_{\mathbb{T}} |p_\nu| \, dm \leqslant \left\{ \int_T |p_\nu|^p \, dm \right\}^{1/p} \to \|f\|_p.$$

Since $\log |p_\nu| \to -\infty$ uniformly on B, the contradiction now follows.

It remains to prove that (iii) $\Rightarrow$ (iv). The proof is a modification of ideas used by Davie, Gamelin, and Garnett in their study [2] of pointwise bounded approximation and distance estimates for rational functions. Since our situation is somewhat different, we give the proof in detail.

Let L denote the space of all pairs $\tilde{f} = (f_1, f_2)$ of functions $f_1 \in L^p(\mathbb{T})$, $f_2 \in C(\bar{F})$. With the norm

$$\|\tilde{f}\| = \max\{\|f_1\|_p, \|f_2\|_{\bar{F}}\},$$

L is a Banach space. If $f \in H^p$ and $f|_F$ is uniformly continuous, we can write (by abuse of notation)

$$f = (f|_{\mathbb{T}}, f|_{\bar{F}}) \in L.$$

Choose $f_0 \in H^p$ and $g \in C(\bar{F})$. such that $\|f_0\|_p = 1$ and $\|f_0 - g\|_F = \delta$, say, where $\delta < \infty$. If $\varepsilon > 0$ is arbitrary, define

$$K_\varepsilon = \{(f_1, f_2) \in L : \|f_1\|_p < 1 + \varepsilon, \|f_2\|_F < \delta + \varepsilon\}.$$

Let J be a compact subset of F—we may suppose that J is infinite. We must show that $f_0|_J$ can be uniformly approximated on J by elements from $P \cap K_\varepsilon$. By arguing as in [2, pp. 45ff], we choose a measure λ on J such that $\operatorname{Re} \lambda(p) \leqslant 1$ for all polynomials $p \in P \cap K_\varepsilon$ and must show that $\operatorname{Re} \lambda(f_0) \leqslant 1$. By Lemma 4.1 in [2], the functional $p \to \lambda(p)$ can be extended to a functional $\mathscr{L} = (\mathscr{L}_1, \mathscr{L}_2)$ on L satisfying $\operatorname{Re} \mathscr{L}(f) \leqslant a$ for all $f \in K_\varepsilon$. Here $\mathscr{L}_1 \in L^q(\mathbb{T})$, $(1/p) + (1/q) = 1$; and $\mathscr{L}_2$ is a measure on $\bar{F}$. Exactly as in [8, p. 305], we show that the restriction $\mathscr{L}_2|_\mathbb{T}$ of $\mathscr{L}_2$ to $\mathbb{T}$ is absolutely continuous with respect to dm.

Thus, $\mathscr{L}(f_0)$ is well-defined, even though f_0 may be outside L.

It is now sufficient to establish the following:

(γ) $\lambda(f_0) = \mathscr{L}(f_0)$,

(δ) $\operatorname{Re} \mathscr{L}(f_0) \leqslant 1$.

Let f_0 have the inner–outer factorization (see [4]) $f_0 = I_0 F_0$, where

$$F_0(z) = \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} v(\theta)\, d\theta.$$

If we replace v by $v_n = \min\{v, n\}$ in this expression, we get a bounded analytic function F_n such that $\|F_n\|_p \leqslant \|F_0\|_p$ and such that $F_n \to F_0$ in m-measure on $\mathbb{T}$ as $n \to \infty$. By the above remarks about $\mathscr{L}_2|_\mathbb{T}$, it is sufficient for part (γ) to verify that (γ) holds with f_0 replaced by $f_n = I_0 F_n$. Let $g_{n,k}(z) = f_n(r_k z)$, where $r_k = 1 - (1/k)$. Since $\mathscr{L}$ is an extension of λ, and since $(\mathscr{L} - \lambda) \perp P$, it follows that

$$\lambda(g_{n,k}) - L(g_{n,k}) = 0, \qquad k = 1, 2,...,$$

because $g_{n,k}$ is a uniform limit of polynomials. The proof that (γ) holds is now completed by letting $k \to \infty$.

Finally we have to prove (δ) for f_0. By our hypothesis and Fatou's theorem we have $|f_0 - g| \leqslant \delta$ almost everywhere on $\bar{F} \cap \mathbb{T}$. So we can find continuous functions G_ν on $\bar{F}$ with $|G_\nu| \leqslant \delta$, $\nu = 1, 2,...$, such that $G_\nu \to f_0 - g$ pointwise on F and in $L^p(dm)$-norm on $\bar{F} \cap \mathbb{T}$. This is where the hypothesis (iii) is used. We extend $\{G_\nu\}$ to $\bar{F} \cup \mathbb{T}$ so that

$$\int_\mathbb{T} |G_\nu - (f_0 - g)|^p \, dm \to 0.$$

Therefore by the dominated convergence theorem

$$\mathscr{L}(f_0) = \mathscr{L}_1(f_0) + \mathscr{L}_2(f_0) = \int_\mathbb{T} f_0 l_1 \, dm + \int_{\bar{F}} f_0 \, d\mathscr{L}_2$$

$$= \lim_{\nu \to \infty} \mathscr{L}(g + G_\nu).$$

Since $(g + G_\nu) \in \bigcap_{\varepsilon > 0} K_\varepsilon$ for $\nu = 1, 2, \ldots$, it follows that $\operatorname{Re} \mathscr{L}(f_0) \leqslant 1$, and the proof of our theorem is complete.

4. Open Questions

(1) What are the results that correspond to our theorem when τ_p is taken as the norm topology on H^p?

(2) What about other spaces like BMOA, the Bergmann spaces, the Bloch space, and so on?

We would finally like to point out a unifying condition necessary and sufficient for F to be a Farrell set for A, when $A = H^p$ or $A = H(D)$, the space of all holomorphic functions in D.

Let $A_\eta = \{f \in A : |f| \leqslant \eta \text{ on } F\}$, where $0 < \eta < 1$. If $A = H^p$, let V denote the unit ball in A. If $A = H(D)$, let $V = V_K = \{f \in A : |f| \leqslant 1 \text{ on } K\}$, where K is a compact subset of K.

Then form the following real valued functions:

$$M(z) = \sup\{|f(z)| : f \in A_\eta \cap V\},$$

$$m(z) = \sup\{|p(z)| : p \in P \cap A_\eta \cap V\},$$

where P denotes the polynomials.

One can then observe from known results that F is a Farrell set for A if and only if $M \equiv m$ for all $\eta \in (0, 1)$.

Possibly this is a condition which may be useful in classifying Farrell and Mergelyan sets for space where a simple geometric description is unlikely.

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